

THE CALDERÓN-ZYGMUND INEQUALITY AND SOBOLEV SPACES ON NONCOMPACT RIEMANNIAN MANIFOLDS

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ABSTRACT. We introduce the concept of Calderón-Zygmund inequalities on Riemannian manifolds. For $1 < p < \infty$, these are inequalities of the form

$$\|\text{Hess}(u)\|_{\mathbf{L}^p} \leq C_1 \|u\|_{\mathbf{L}^p} + C_2 \|\Delta u\|_{\mathbf{L}^p},$$

valid a priori for all smooth functions u with compact support, and constants $C_1 \geq 0$, $C_2 > 0$. Such an inequality can hold or fail, depending on the underlying Riemannian geometry. After establishing some generally valid facts and consequences of the Calderón-Zygmund inequality (like new denseness results for second order \mathbf{L}^p -Sobolev spaces and gradient estimates), we establish sufficient geometric criteria for the validity of these inequalities on possibly noncompact Riemannian manifolds. These results in particular apply to many noncompact hypersurfaces of constant mean curvature.

1. INTRODUCTION

Let M be a smooth possibly noncompact Riemannian manifold. For an arbitrary $p \in (1, \infty)$, let us consider the following canonically given problems for second order Sobolev spaces on M , on the \mathbf{L}^p -scale:

- *Problem 1: Under which (geometric) assumptions on M does one have the denseness $\mathbf{H}_0^{2,p}(M) = \mathbf{H}^{2,p}(M)$?*
- *Problem 2: Under which assumptions on M does one have the implication*

$$f \in \mathbf{L}^p(M) \cap \mathbf{C}^2(M), \Delta f \in \mathbf{L}^p(M) \Rightarrow f \in \mathbf{H}^{2,p}(M)$$

(that is, $|\text{Hess}(f)| \in \mathbf{L}^p(M)$)?

- *Problem 3: Under which assumptions on M does one have an inequality of the form*

$$(1) \quad \|\text{grad}(f)\|_{\mathbf{L}^p} \leq C(\|\Delta f\|_{\mathbf{L}^p} + \|f\|_{\mathbf{L}^p}) \quad \text{for all } f \in \mathbf{C}_c^\infty(M)?$$

2010 *Mathematics Subject Classification.* 53C20, 46E35, 58J50.

Let us note here that Problem 1, the denseness of $C_c^\infty(M)$ in $H^{2,p}(M)$, is a classical problem which has been treated systematically in [21]. Here, we would like to stress the fact that without a lower control on the injectivity results, nothing seems to be known so far for the case $p \neq 2$. Furthermore, Problem 2 is obviously concerned with L^p -estimates for solutions of the Poisson equation on M , and Problem 3 arises naturally in Rellich-Kondrachov type compactness arguments: Here, typically one has given a sequence of functions $\{f_n\} \subset C_c^\infty(M)$ such that

$$\sup_{n \in \mathbb{N}} \max(\|f_n\|, \|\Delta f_n\|) < \infty,$$

and one would like to know whether the sequence $\{f_n\}$ is bounded in $H^{1,p}(M)$.

It turns that there is an inequality underlying all three problems simultaneously, namely, the *Calderón-Zygmund inequality*. This inequality, which may generally fail on noncompact M 's, states that there are constants $C_1 \geq 0$, $C_2 > 0$ such that

$$(CZ(p)) \quad \|\text{Hess}(u)\|_{L^p} \leq C_1 \|u\|_{L^p} + C_2 \|\Delta u\|_{L^p} \quad \text{for all } u \in C_c^\infty(M).$$

Let us remark that in the Euclidean \mathbb{R}^m , this inequality can be proved [16] by estimates on singular integral operators which have been proved by Calderón and Zygmund [5]. Ultimately, this was the motivation for us for calling¹ $CZ(p)$ the “Calderón-Zygmund inequality”.

Now one has the following implications which link $CZ(p)$ to the above Problems 1-3:

- (A) It has been observed in [19] *that under $CZ(p)$, if M is geodesically complete and admits a sequence of Laplacian cut-off functions (this is the case e.g. if M has a nonnegative Ricci curvature), then one has $H_0^{2,p}(M) = H^{2,p}(M)$.*
- (B) In Corollary 3.10 *we show that under $CZ(p)$, if M is geodesically complete and admits a sequence of Hessian cut-off functions (this is the case e.g. if M has a bounded curvature tensor), then any $f \in H^{1,p}(M) \cap C^2(M)$ with $\Delta f \in L^p(M)$ satisfies $f \in H^{2,p}(M)$.*
- (C) In Corollary 3.13 *we prove that, on a geodesically complete manifold, $CZ(p)$ always implies (1).*

Here, the results (A) and (B) follow from the existence of appropriate second order cut-off functions (see also [19]), which we prove to exist under very weak assumptions on the curvature and without positive

¹The authors would like to thank Klaus Ecker in this context

injectivity radius. In this context, we also establish our first main result (cf. Corollary 3.9 below):

Theorem 1.1. *Let M be geodesically complete with a bounded curvature tensor. Then one has $H_0^{2,p}(M) = H^{2,p}(M)$ for all $1 < p < \infty$.*

This result is entirely new for $p \neq 2$, for it does not require a positive injectivity radius (cf. [21]).

The statement (C) makes use of an appropriate L^p -interpolation result, which should be of an independent interest (cf. Proposition 3.12). These observations clearly motivate a systematic treatment of the following problem:

Under which (geometric) assumptions on M does one have $CZ(p)$, and how do the $CZ(p)$ -constants C_1, C_2 depend on the underlying geometry?

Let us start by taking a look at the *local situation*: We first prove in Theorem 3.14 that one always has $CZ(p)$ on relatively compact domains $\Omega \subset M$, where one can even pick $C_1 = 0$, if Ω has a smooth boundary. In particular, using a gluing procedure which again relies on L^p -interpolation, the latter results show that $CZ(p)$ is stable under compact perturbations (cf. Theorem 3.14), which in particular applies to manifolds with ends. However, as one might expect, in both of these cases the $CZ(p)$ -constants depend rather implicitly on the underlying geometry, which raises the question of more precise estimates on geodesic balls: This problem is attacked in Theorem 3.16, where we prove that $CZ(p)$ holds on sufficiently small geodesic balls, with constants only depending on the radius, $\dim M$, p , and a lower bound of an appropriate harmonic radius.

As for *global results*, it turns out that for $p = 2$ it is possible to give a rather complete answer: Namely, it is shown in Proposition 4.1 that a lower bound $\text{Ric} \geq -C$ on the Ricci curvature implies a stronger infinitesimal variant of $CZ(2)$ with constants depending explicitly on C . This is in fact a straightforward consequence of Bochner's identity. On the other hand, we prove that this result is optimal, in the sense that there exists a geodesically complete noncompact surface N with unbounded Gauss curvature, such that $CZ(2)$ fails on N (cf. Theorem 4.2).

For $p \neq 2$, we prove the following two results, which can also be considered as the main results of this paper:

Theorem 1.2. *Let $1 < p < \infty$ and assume that M has bounded Ricci curvature and a positive injectivity radius². Then one has $CZ(p)$, with*

²thus M is automatically geodesically complete

constants depending only on $\dim M$, p , $\|\text{Ric}\|_\infty$ and the injectivity radius.

Theorem 1.3. *Let $1 < p \leq 2$. Assume that M is geodesically complete with a first order bounded geometry, and that there are constants $D \geq 1$, $0 \leq \delta < 2$ with*

$$(2) \quad \text{vol}(B_{tr}(x)) \leq Dt^D e^{t^\delta + r^\delta} \text{vol}(B_r(x)) \quad \text{for all } x \in M, r > 0, t \geq 1.$$

Then one has CZ(p), with constants depending only on $\dim M$, p , $\|\text{R}\|_\infty$, $\|\nabla \text{R}\|_\infty$, D, δ , with R the curvature tensor.

The proof of Theorem 1.2 uses appropriate elliptic estimates, in combination with harmonic-radius bounds on the Riemannian structure, a method which requires bounds on the injectivity radius, but has the advantage of working for arbitrary p . The proof of Theorem 1.3, however, is very different: It uses deep boundedness-results on *covariant Riesz-transforms* by Thalmaier-Wang [30], which ultimately follow from covariant probabilistic heat-semigroup derivative formulas. This technique makes it possible to avoid assumptions on the injectivity radius. Under geodesic completeness, the generalized doubling assumption (2) is implied by $\text{Ric} \geq 0$ (though $\text{Ric} \geq 0$ is not necessary at for all (2); cf. Example 4.7).

This paper is organized as follows: In Section 2 we establish some Riemann geometric notation. In Section 3 we first establish the above mentioned consequences (A), (B), (C) of the Calderón-Zygmund inequality, and then we prove the various local Calderón-Zygmund inequalities (Theorem 3.14 and Theorem 3.16). In Section 4 we prove several geometric criteria for the validity CZ(p) on noncompact M 's, thus Proposition 4.1 for $p = 2$, and the above Theorem 1.2 and Theorem 1.3, and Section 5 is devoted to the construction of the surface which does not support CZ(2) fails. Finally, in Section 6 we apply Theorem 1.2 to noncompact hypersurfaces of constant mean curvature. We have also included two appendices, where some facts on harmonic-radius bounds and on abstract Riemannian gluings have been collected for the convenience of the reader.

2. SETTING AND NOTATION

We fix an arbitrary smooth Riemannian m -manifold $M \equiv (M, g)$ ³ with ∇ the Levi-Civita connection on M . We will denote the corresponding distance function with $d(\bullet, \bullet)$, the open balls with $B_a(x)$, $x \in M$, $a > 0$,

³In the sequel a manifold will always be understood to be without boundary, unless otherwise stated

and the volume measure with $\mu(dx) := \text{vol}(dx)$, where whenever there is no danger of confusion we shall simply write $\int f d\mu$ instead of $\int_M f d\mu$. The symbol $r_{\text{inj}}(x) \in (0, \infty]$ will stand for the injectivity radius at x , with

$$r_{\text{inj}}(M) := \inf_{x \in M} r_{\text{inj}}(x) \in [0, \infty]$$

the global injectivity radius.

Let us develop some further geometric notation which will be used in the sequel: If $E \rightarrow M$ is a smooth Euclidean vector bundle, then whenever there is no danger of confusion we will denote the underlying Euclidean structure simply with $(\bullet, \bullet)_x$, $x \in M$, $|\bullet|_x := \sqrt{(\bullet, \bullet)_x}$ will stand for the corresponding norm on E_x . Using μ , for any $1 \leq p \leq \infty$ we get the corresponding L^p -spaces of equivalence classes of Borel sections $\Gamma_{L^p}(M, E)$, with their norms

$$\|f\|_p := \begin{cases} \left(\int_M |f(x)|_x^p \mu(dx) \right)^{1/p}, & \text{if } p < \infty \\ \inf\{C | C \geq 0, |f| \leq C \text{ } \mu\text{-a.e.}\}, & \text{if } p = \infty. \end{cases}$$

The symbol $\langle \bullet, \bullet \rangle$ will stand for the canonical inner product on the Hilbert space $\Gamma_{L^2}(M, E)$, and $'\dagger'$ will denote the formal adjoint with respect to $\langle \bullet, \bullet \rangle$ of a smooth linear partial differential operator that acts on some $E \rightarrow M$ as above.

We equip T^*M with its canonical Euclidean structure

$$(\alpha_1, \alpha_2) := g(\alpha_1^\sharp, \alpha_2^\sharp), \quad \alpha_1, \alpha_2 \in \Gamma_{C^\infty}(M, T^*M),$$

where α_j^\sharp stands for the vector field which is defined by α in terms of g . This produces canonical Euclidean metrics on all bundles of k -times contravariant and l -times covariant tensors $T^{k,l}M \rightarrow M$. Next, ∇ induces a Euclidean covariant derivative on $T^*M = T^{0,1} \rightarrow M$ through

$$\nabla_{X_1} \alpha(X_2) := X_1(\alpha(X_2)) - \alpha(\nabla_{X_1} X_2),$$

for any smooth 1-form α and any smooth vector fields X_1, X_2 on M , which of course means nothing but $(\nabla_{X_1} \alpha)^\sharp = \nabla_{X_1} \alpha^\sharp$, and these data are tensored to give a Euclidean covariant derivative on $T^{k,l}M \rightarrow M$ which, by the above abuse of notation, is always denoted with ∇ . Thus, for any $u \in C^\infty(M)$ we have

$$\text{Hess}(u) := \nabla du \in \Gamma_{C^\infty}(M, T^{0,2}).$$

The gradient $\text{grad}(u) \in \Gamma_{C^\infty}(M, TM)$, is defined by

$$(\text{grad}(u), X) := du(X) \text{ for any } X \in \Gamma_{C^\infty}(M, TM),$$

with

$$d : \Gamma_{C^\infty}(M, \wedge^\bullet T^*M) \longrightarrow \Gamma_{C^\infty}(M, \wedge^{\bullet+1} T^*M)$$

the exterior differential, and where as usual 0-forms are identified with functions. Then the divergence $\operatorname{div}(X) \in C^\infty(M)$ of a smooth vector field X on M is given by $\operatorname{div}(X) = d^\dagger X^\flat$, where X^\flat stands for the 1-form which is defined by X in terms of $g = (\bullet, \bullet)$. Let us denote with

$$\Delta_\bullet := d^\dagger d + d d^\dagger : \Gamma_{C^\infty}(M, \wedge^\bullet T^*M) \longrightarrow \Gamma_{C^\infty}(M, \wedge^\bullet T^*M)$$

the Laplace-Beltrami operator on differential forms. Note that our sign convention is such that Δ_\bullet is *nonnegative*, and the Friedrichs realization of Δ_\bullet in $\Gamma_{L^2}(M, \wedge^\bullet T^*M)$ will be denoted with the same symbol. In the sequel, we will freely use the formulas

$$\begin{aligned} \operatorname{Hess}(u_1 u_2) &= u_2 \operatorname{Hess}(u_1) + du_1 \otimes du_2 + du_2 \otimes du_1 + u_1 \operatorname{Hess}(u_2), \\ \Delta u_1 &= -\operatorname{tr}(\operatorname{Hess}(u_1)) = -\operatorname{div}(\operatorname{grad}(u_1)), \\ \Delta(u_1 u_2) &= \Delta(u_1) u_2 - 2(\operatorname{grad}(u_1), \operatorname{grad}(u_2)) + \Delta(u_2) u_1, \\ \operatorname{div}(u_1 X) &= (\operatorname{grad}(u_1), X) + u_1 \operatorname{div}(X), \end{aligned}$$

valid for all smooth functions u_1, u_2 and smooth vector fields X on M . We close this section with some conventions and notation which concerns curvature data: The curvature tensor R is read as a section

$$R \in \Gamma_{C^\infty}(M, T^{1,3}M),$$

given for smooth vector fields X, Y, Z by

$$R(X, Y, Z) := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \in \Gamma_{C^\infty}(M, TM).$$

Then the Ricci curvature

$$\operatorname{Ric} \in \Gamma_{C^\infty}(M, T^{0,2}M)$$

is the section given by the fiberwise trace

$$\operatorname{Ric}(Z, Y) = \operatorname{tr}(X \mapsto R(X, Y, Z)) \in C^\infty(M).$$

If $m \geq 2$, then for any $x \in M$, the sectional curvature of a two dimensional subspace $A = \operatorname{span}(X, Y)$ of $T_x M$ is well-defined by

$$\operatorname{Sec}(A) := \frac{(R(X, Y, Y), X)}{|X \wedge Y|^2} \in \mathbb{R}.$$

Finally, we mention that whenever we write $C = C(a_1, \dots, a_l)$ for a constant, this means that C only depends on the parameters a_1, \dots, a_l , and nothing else.

3. CONSEQUENCES OF THE CALDERÓN-ZYGMUND INEQUALITY AND LOCAL CONSIDERATIONS

In this section, we are going to collect some abstract and fundamental facts on Calderón-Zygmund inequalities.

We start with:

Definition 3.1. *Let $1 < p < \infty$. We say that $M \equiv (M, g)$ satisfies the L^p -Calderón-Zygmund inequality (or in short CZ(p)), if there are $C_1 \geq 0$, $C_2 > 0$, such that for all $u \in C_c^\infty(M)$ one has*

$$(3) \quad \|\text{Hess}(u)\|_p \leq C_1 \|u\|_p + C_2 \|\Delta u\|_p.$$

Obviously, if some CZ(p) holds on a Riemannian manifold M then, by restriction, the same inequality holds on any open subset of M and with the same constants. Furthermore, we have directly excluded the extremal cases $p = 1$ and $p = \infty$ in Definition 3.1 since the corresponding hypothetical elliptic estimates fail [27, 12] for the Euclidean Laplace operator $\sum_i \partial_i^2$, thus (3) with $p = 1$ or $p = \infty$ cannot hold in general.

Let us record that a certain scaling rigidity in the constants implies automatically that one can pick $C_1 = 0$ (noting that in Riemannian geometry, such a stability typically appears in the context of nonnegative Ricci curvature):

Remark 3.2. *1. Let $1 < p < \infty$, and assume that there are $C_1 \geq 0$, $C_2 > 0$ such that for all $0 < \lambda \leq 1$ one has (3) with respect to the Riemannian metric $\lambda^2 g$. Then one has (3) with $C_1 = 0$ (with respect to g). Indeed, the assumption implies*

$$\|\text{Hess}(u)\|_p \leq C_1 \lambda^{2p} \|u\|_p + C_2 \|\Delta u\|_p,$$

with respect to g with C_j uniform in λ , and we can take $\lambda \rightarrow 0+$.

2. As a particular case of the above situation, assume $\text{Ric} \geq 0$, $1 < p < \infty$ and that there are $C_1 \geq 0$, $C_2 > 0$, which do not depend on g , such that one has (3). Then one has (3) with $C_1 = 0$.

We continue with some important consequences of the Calderón-Zygmund inequalities. Let us start with some remarks concerning the connection between CZ(p) and second order L^p -Sobolev spaces. In fact, precisely this context was the original motivation for our study of Calderón-Zygmund inequalities. To this end, we first list some conventions and notation on Sobolev spaces: For any $1 \leq p < \infty$, the Banach space $H^{1,p}(M)$ is defined by

$$H^{1,p}(M) := \{u \mid u \in L^p(M), |\text{grad}(u)| \in L^p(M) \text{ as distr.}\},$$

with its natural norm

$$\|u\|_{1,p} := \|u\|_p + \|\text{grad}(u)\|_p.$$

Likewise, one has the Banach space

$$\mathbf{H}^{2,p}(M) := \{u \mid u \in \mathbf{L}^p(M), |\text{grad}(u)|, |\text{Hess}(u)| \in \mathbf{L}^p(M) \text{ as distr.}\},$$

with its natural norm $\|u\|_{2,p}$. By a generalized Meyers-Serrin type theorem [18], one has that the linear space

$$\mathbf{C}^\infty(M) \cap \mathbf{H}^{k,p}(M) \text{ is dense in } \mathbf{H}^{k,p}(M)$$

(a fact which is actually true for all $k \in \mathbb{N}$ with the natural definition of higher order Sobolev spaces). Finally, we define $\mathbf{H}_0^{k,p}(M) \subset \mathbf{H}^{k,p}(M)$ as usual to be the closure of $\mathbf{C}_c^\infty(M)$ in $\mathbf{H}^{k,p}(M)$.

Remark 3.3. *Let $1 \leq p < \infty$.*

1. Every $u \in \mathbf{H}^{2,p}(M)$ satisfies $\Delta u \in \mathbf{L}^p(M)$ in the sense of distributions. Indeed, integrating by parts and using

$$\text{Hess}^\dagger \circ \text{tr}^\dagger = (\text{tr} \circ \text{Hess})^\dagger,$$

where we consider $\text{tr}(\bullet)$ as a smooth zeroth order linear differential operator, one gets that the distribution Δu is in fact a Borel function which coincides with

$$\Delta u(x) = -\text{tr}_x(\text{Hess}(u)|_x),$$

so that

$$|\Delta u| \leq \sqrt{m} |\text{Hess}(u)| \quad \mu\text{-a.e. in } M.$$

2. If $u \in \mathbf{H}_0^{2,p}(M)$, and if $\{u_k\} \subset \mathbf{C}_c^\infty(M)$ is a sequence such that $\|u - u_k\|_{2,p} \rightarrow 0$, then obviously $\{u_k\}$ is Cauchy in $\mathbf{L}^p(M)$, $\{\text{Hess}(u_k)\}$ is Cauchy in $\Gamma_{\mathbf{L}^p}(M, \mathbf{T}^{0,2}M)$, and using

$$(4) \quad |\Delta \psi| \leq \sqrt{m} |\text{Hess}(\psi)|, \quad \text{for all } \psi \in \mathbf{C}^\infty(M),$$

it also follows that $\{\Delta u_k\}$ is Cauchy in $\mathbf{L}^p(M)$. In particular, one necessarily has

$$\|u - u_k\|_p \rightarrow 0, \quad \|\text{Hess}(u) - \text{Hess}(u_k)\|_p \rightarrow 0, \quad \|\Delta u - \Delta u_k\|_p \rightarrow 0.$$

Remark 3.3.2 immediately implies that Calderón-Zygmund inequalities always extend to $\mathbf{H}_0^{2,p}(M)$ in the following sense:

Corollary 3.4. *Let $1 < p < \infty$. If one has (3), then this inequality extends from $\mathbf{C}_c^\infty(M)$ to $\mathbf{H}_0^{2,p}(M)$ with the same constants.*

The following definition will be convenient (cf. [19]):

Definition 3.5. a) *M is said to admit a sequence $(\chi_n) \subset \mathbf{C}_c^\infty(M)$ of Laplacian cut-off functions, if (χ_n) has the following properties:*

- (C1) $0 \leq \chi_n(x) \leq 1$ for all $n \in \mathbb{N}$, $x \in M$,
- (C2) for all compact $K \subset M$, there is an $n_0(K) \in \mathbb{N}$ such that for all $n \geq n_0(K)$ one has $\chi_n|_K = 1$,
- (C3) $\sup_{x \in M} |d\chi_n(x)|_x \rightarrow 0$ as $n \rightarrow \infty$,
- (C4) $\sup_{x \in M} |\Delta\chi_n(x)| \rightarrow 0$ as $n \rightarrow \infty$.

b) M is said to admit a sequence $(\chi_n) \subset C_c^\infty(M)$ of Hessian cut-off functions, if (χ_n) has the above properties (C1), (C2), (C3), and in addition

$$(C4') \sup_{x \in M} |\text{Hess}(\chi_n)(x)|_x \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Remark 3.6. By (4), any sequence of Hessian cut-off functions is automatically a sequence of Laplacian cut-off functions.

One has:

Proposition 3.7. a) If M is geodesically complete with $\text{Ric} \geq 0$, then M admits a sequence of Laplacian cut-off functions.

b) If M is geodesically complete with $\|\mathbf{R}\|_\infty < \infty$, then M admits a sequence of Hessian cut-off functions.

Proof. a) This result is included in [19]. It relies on a rigidity result by Cheeger and Colding [7].

b) Let $m = \dim M$. By a result of L.-F. Tam⁴, see Proposition 26.49 in [8], one has that there is a constant $C = C(\|\mathbf{R}\|_\infty, m) > 0$, such that for any $x_0 \in M$ there is a smooth function $\tilde{d} = \tilde{d}_{x_0} : M \rightarrow [0, \infty)$ satisfying

$$(5) \quad d(\bullet, x_0) + 1 \leq \tilde{d} \leq d(\bullet, x_0) + C, \quad |\text{grad}(\tilde{d})| \leq C, \quad |\text{Hess}(\tilde{d})| \leq C.$$

Pick now a smooth function $t : \mathbb{R} \rightarrow [0, 1]$ which is compactly supported, equal to 1 in $[0, C + \frac{1}{2}]$, and zero on $[C + 1, \infty)$. Then $\chi_n(x) := t(\tilde{d}(x)/n)$ has the required properties. \square

Proposition 3.8. a) Assume that (3) holds for some $1 < p < \infty$ and that M admits a sequence of Laplacian cut-off functions. Then one has $H_0^{2,p}(M) = H^{2,p}(M)$, in particular (by Corollary 3.4), (3) extends to $H^{2,p}(M)$ with the same constants.

b) If M admits a sequence of Hessian cut-off functions, then one has $H_0^{2,p}(M) = H^{2,p}(M)$ for all $1 < p < \infty$.

Proof. Part a) has been observed in [19]. Part b) follows from the same argument: Given a smooth $f \in H^{2,p}(M)$, pick a sequence (χ_n) of

⁴ a completely different construction of an exhaustion function with bounded gradient and Hessian is also contained in [10].

Hessian cut-off functions. With $f_n := \chi_n f$, using

$$\text{Hess}(f_n) = f \text{Hess}(\chi_n) + d\chi_n \otimes df + df \otimes d\chi_n + \chi_n \text{Hess}(f),$$

one easily gets that f_n converges to f in $H^{2,p}(M)$. \square

We immediately get:

Corollary 3.9. *If M is geodesically complete with $\|R\|_\infty < \infty$, then one has $H_0^{2,p}(M) = H^{2,p}(M)$ for all $1 < p < \infty$.*

Let us continue with a connection between Calderón-Zygmund inequalities and global control of solutions to the Poisson equation: Classically, one uses *local* L^p -Calderón-Zygmund type inequalities in order to get higher *local* regularity of solutions of the Poisson equation $\Delta u = f$. Indeed, if u is in $W_{\text{loc}}^{1,p}$ and f is in L_{loc}^p then a local Calderón-Zygmund type inequality shows $\partial_i \partial_j u$ is in L_{loc}^p , proving that u is in $W_{\text{loc}}^{2,p}$. In fact, one can use this way of concluding to derive global estimates to solutions of the Poisson equation:

Proposition 3.10. *Let $1 < p < \infty$, and assume that M satisfies CZ(p) and admits a sequence of Hessian cut-off functions. Let $u \in C^2(M)$ be a solution of the Poisson equation*

$$\Delta u = f.$$

If $u, |\text{grad}(u)|, f \in L^p(M)$ then $|\text{Hess}(u)| \in L^p(M)$.

Remark 3.11. *It is not completely clear to what extent the L^p assumption on the gradient is technical and related to the method of proof. In any case, it would be interesting to find situations where it is automatically satisfied. Compare also with Corollary 3.13.*

Proof of Proposition 3.10. Pick a sequence of Hessian cut-off functions (φ_k) and defining the corresponding sequence of compactly supported C^2 -functions $u_k = u\varphi_k$. Then applying to u_k Calderón-Zygmund inequality we obtain (using Corollary 3.4)

$$\begin{aligned} & \|\varphi_k \text{Hess}(u)\|_p \\ & \leq C \left(\|u \text{Hess}(\varphi_k)\|_p + \| |\text{grad}(u)| |\text{grad}(\varphi_k)| \|_p + \|u \Delta \varphi_k\|_p + \|\varphi_k \Delta u\|_p \right). \end{aligned}$$

Whence, taking the limit as $k \rightarrow \infty$, the claim follows from dominated convergence. \square

In order to prove our next application of CZ(p), a gradient estimate, we will need the following L^p -interpolation result, which should be of an independent interest, and which will also be used later on to prove local CZ(p) inequalities:

Proposition 3.12. a) *For any $2 \leq p < \infty$ there is a constant $C = C(p) > 0$ such that for any $\varepsilon > 0$, $u \in \mathcal{C}_c^\infty(M)$ one has*

$$(6) \quad \|\text{grad}(u)\|_p \leq \frac{C}{\varepsilon} \|u\|_p + C\varepsilon \|\text{Hess}(u)\|_p.$$

b) *Assume that either M is geodesically complete or that M is a relatively compact open subset of an arbitrary smooth Riemannian manifold. Then for any $1 < p \leq 2$ there is a constant $C = C(p) > 0$ such that for any $\varepsilon > 0$, $u \in \mathcal{C}_c^\infty(M)$ one has*

$$(7) \quad \|\text{grad}(u)\|_p \leq \frac{C}{\varepsilon} \|u\|_p + C\varepsilon \|\Delta u\|_p.$$

Proof. a) Let $u \in \mathcal{C}_c^\infty(M)$ and, having fixed $\alpha > 0$, consider the smooth, compactly supported vector field

$$X := u \cdot (|\text{grad}(u)|^2 + \alpha)^{\frac{p-2}{2}} \text{grad}(u).$$

Using the divergence theorem and elaborating, we obtain

$$\begin{aligned} & \int (|\text{grad}(u)|^2 + \alpha)^{\frac{p-2}{2}} |\text{grad}(u)|^2 \, d\mu \\ & \leq |p-2| \int |u| (|\text{grad}(u)|^2 + \alpha)^{\frac{p-4}{2}} |\text{grad}(u)|^2 |\text{Hess}(u)| \, d\mu \\ & \quad + \int |u| |\Delta u| (|\text{grad}(u)|^2 + \alpha)^{\frac{p-2}{2}} \, d\mu. \end{aligned}$$

Letting $\alpha \rightarrow 0$ and applying the monotone and dominated convergence theorems we get⁵

$$\begin{aligned} (8) \quad & \int |\text{grad}(u)|^p \, d\mu \\ & \leq |p-2| \int |u| |\text{Hess}(u)| |\text{grad}(u)|^{p-2} \, d\mu + \int |u| |\Delta u| |\text{grad}(u)|^{p-2} \, d\mu. \end{aligned}$$

Now, in both the integrands appearing in the right hand side of (8), we use the Young inequality

$$ab \leq \frac{1}{\varepsilon^{p'} p'} a^{p'} + \frac{\varepsilon^{q'}}{q'} b^{q'}, \quad a, b \geq 0$$

with

$$p' = p/2, \quad q' = p/(p-2).$$

⁵obviously, by monotone convergence, the same integral inequality holds if $p < 2$. However, in this case, the right hand side could be infinite. For instance, in \mathbb{R}^2 , this happens if $p = 1$ as one can see by taking $u(x, y) = (x^2 + 1)\varphi(x, y)$, where $0 \leq \varphi \leq 1$ is a cut-off function satisfying $\varphi = 1$ on $[0, 1] \times [0, 1]$.

We obtain that the right-hand side of (8) is

$$(9) \quad \leq \frac{1}{\varepsilon^{p'} p'} \int |u|^{\frac{p}{2}} |\text{Hess}(u)|^{\frac{p}{2}} d\mu + \frac{1}{\varepsilon^{p'} p'} \int |u|^{\frac{p}{2}} |\Delta u|^{\frac{p}{2}} d\mu + 2 \frac{\varepsilon^{q'}}{q'} \int |\text{grad}(u)|^p.$$

Choose $0 < \varepsilon = \varepsilon(p) < 1$ so small that

$$A_p := 1 - 2(p-2) \frac{\varepsilon^{q'}}{q'} > 0,$$

and let $B_p := 1/(\varepsilon^{p'} p')$. Then, inserting (9) into (8) we deduce

$$\begin{aligned} & A_p \int |\text{grad}(u)|^p d\mu \\ & \leq (p-2) B_p \int |u|^{\frac{p}{2}} |\text{Hess}(u)|^{\frac{p}{2}} d\mu + B_p \int |u|^{\frac{p}{2}} |\Delta u|^{\frac{p}{2}} d\mu. \end{aligned}$$

Whence, using twice the Young inequality

$$(10) \quad ab \leq \frac{1}{2\varepsilon^2} a^2 + \frac{\varepsilon^2}{2} b^2$$

with any arbitrary $\varepsilon > 0$ we conclude

$$\begin{aligned} & \int |\text{grad}(u)|^p d\mu \\ & \leq \varepsilon^{-2} C_p \int |u|^p d\mu + \varepsilon^2 D_p \left(\int |\text{Hess}(u)|^p d\mu + \int |\Delta u|^p d\mu \right) \end{aligned}$$

where we have set

$$C_p := \frac{(p-1) B_p}{2A_p}, \quad D_p := \frac{\max(p-2, 1) B_p}{2A_p}.$$

b) Assume first that M is geodesically complete. Then by Theorem 4.1 of [9] we have the multiplicative inequality

$$(11) \quad \|\text{grad}(u)\|_p \leq C(p) \|u\|_p^{\frac{1}{2}} \|\Delta u\|_p^{\frac{1}{2}},$$

which completes the proof in this case, using once more inequality (10).

Assume now that M is a relatively compact open subset of an arbitrary smooth and geodesically incomplete Riemannian manifold (\overline{M}, g) . Then as above it is sufficient to prove that (11) remains valid on M , which can be seen, for instance, from the usual construction of complete metrics in a given conformal class: Assume that \overline{M} is an incomplete open manifold, otherwise the conclusion is trivial by restriction. We consider a smooth, relatively compact exhaustion $M_k \subset\subset M_{k+1} \nearrow \overline{M}$ such that $M \subset\subset M_1$ and we pick any smooth function $\lambda : \overline{M} \rightarrow \mathbb{R}_{\geq 0}$ with the following properties: (i) $\lambda(x) = 0$ on M_1 ; (ii) for every $k \geq 1$,

$\lambda(x) = c_k$ on $M_{2k+1} \setminus M_{2k}$, where a sequence $\{c_k\} \subset (0, \infty)$ with $c_k \nearrow \infty$ will be specified later. Next, we define a new metric on \overline{M} by $g_\lambda := e^{2\lambda}g$. By construction, $g_\lambda = g$ on M . Let $r_k = \text{dist}_g(\partial M_{2k+1}, \partial M_{2k})$ and choose $\{c_k\}$ in such a way that $\sum r_k \cdot e^{c_k} = +\infty$. Then, g_λ is geodesically complete. Indeed, if $\gamma : [0, \infty) \rightarrow \overline{M}$ is a divergent path, γ is forced to pass across every annulus $M_{2k+1} \setminus M_{2k}$. Therefore, its length satisfies $\ell(\gamma) \geq \sum r_k \cdot e^{c_k} = \infty$ and this characterizes the geodesic completeness. Since (11) holds on $(\overline{M}, g_\lambda)$, by restriction it holds on its open subset set M , as claimed.

Another way to prove the validity of (11) on relatively compact open subset of possibly incomplete smooth Riemannian manifold is to use the (less elementary) double construction as explained in the proof of Theorem 3.14 a). \square

We immediately get the following Corollary, a gradient estimate which is a variant of (9) for arbitrary p , and which is useful in establishing compactness results for solutions of the Dirichlet problem for the Poisson equation (cf. the proof of Theorem 3.14 a) below):

Corollary 3.13. *In the situation of Proposition 3.12, assume that CZ(p) holds on M for some $1 < p < \infty$. Then, there exists a constant $C > 0$, which only depends on the CZ(p)-constants and p , such that the following inequality*

$$\|\text{grad}(u)\|_p \leq C(\|\Delta u\|_p + \|u\|_p)$$

holds for every $u \in \mathcal{C}_c^\infty(M)$.

The rest of this section is devoted to local aspects of CZ(p) inequalities. We start with the following result, where it is claimed that CZ(p) with $1 < p < \infty$ always holds on relatively compact open subsets (where of course the corresponding constants cannot be controlled explicitly), and moreover that CZ(p) is stable under compact perturbations:

Theorem 3.14. *Let $1 < p < \infty$.*

a) CZ(p) holds on any relatively compact open subset $\Omega \subset M$. Moreover, if $\Omega \subset M$ is a relatively compact domain with smooth boundary $\partial\Omega$, then CZ(p) holds in the stronger form

$$\|\text{Hess}(u)\|_p \leq C \|\Delta u\|_p,$$

for every $u \in \mathcal{C}_c^\infty(\Omega)$ and for a constant $C > 0$. In both cases, the constants depend quite implicitly on the geometry of Ω (cf. the proof).

b) Assume that either $p \geq 2$ or that $1 < p < 2$ and M is geodesically complete. If there is a relatively compact open subset $\Omega \subset M$ such that CZ(p) holds on $M \setminus \overline{\Omega}$, then CZ(p) also holds on M . Here, a

possible choice of Calderón-Zygmund constants on M depends on those of $M \setminus \overline{\Omega}$, those of a relatively compact open neighborhood of Ω , and the choice of a gluing function (cf. the proof for the details).

In the next Corollary we essentially rephrase part b) of Theorem 3.14 in more geometric terms. This formulation involves two geometric objects: (1) the connected sum of Riemannian manifolds, whose construction will be recalled in Appendix B, and (2) the notion of an *end* E of a complete Riemannian manifold M with respect to a compact domain Ω : E is any of the unbounded connected components of $M \setminus \overline{\Omega}$.

Corollary 3.15. *A complete Riemannian manifold supports $\text{CZ}(p)$, $1 < p < \infty$, if and only if each of its ends E_1, \dots, E_k , with respect to any smooth, compact domain Ω , supports the same Calderón-Zygmund inequality. In particular, $\text{CZ}(p)$ holds on the Riemannian connected sum $M = M_1 \# M_2$ of m -dimensional complete Riemannian manifolds M_1 and M_2 if and only if the same inequality holds on both M_1 and M_2 .*

The proof of part a) of Theorem 3.14 relies on the validity of the corresponding $\text{CZ}(p)$ on a closed Riemannian manifold. This reduction procedure is obtained by using the Riemannian double of a manifold with boundary; see Appendix B. See also Remark 4.9 for a different and somewhat more direct argument. On the other hand, for the proof of part b), we will again need the L^p -interpolation inequality from Proposition 3.12 a), which make gluing methods accessible to Calderón-Zygmund inequalities at all.

Proof of Theorem 3.14. a) Let N be a relatively compact domain of M such that ∂N is a smooth hypersurface and $\overline{\Omega} \subset N$. “The” Riemannian double $\mathcal{D}(N)$ of N is a compact Riemannian manifold without boundary. Moreover, by its very construction, it is always possible to assume that $\mathcal{D}(N)$ contains an isometric copy Ω_N of the original domain Ω ; see Appendix B. We shall see in Theorem 4.3 below that every closed manifold supports $\text{CZ}(p)$. In particular, this applies to $\mathcal{D}(N)$, namely, there exist suitable constants $C_1 > 0$ and $C_2 \geq 0$, depending on the geometry of $\mathcal{D}(N)$, such that

$$(12) \quad \|\text{Hess}(u)\|_p \leq C_1 \|\Delta u\|_p + C_2 \|u\|_p,$$

for every $u \in C^\infty(\mathcal{D}(N))$. In particular, the same inequality holds for every $u \in C_c^\infty(\Omega_N)$. Since, up to Riemannian isometries, Ω_N is just the original domain Ω , we conclude that (12) (with the same constants) holds on Ω , as required.

We now assume that Ω as above has smooth boundary and is connected.

Then, in spirit of the proof of Lemma 9.17 in [16] we obtain that there exists a constant $C_3 > 0$ such that

$$(13) \quad \|u\|_p \leq C_3 \|\Delta u\|_p,$$

for every $u \in C_c^\infty(\Omega)$. Inserting this latter into (12) concludes the proof of part a). For the sake of completeness, let us provide a self-contained proof of (13). By contradiction, suppose that there exists a sequence $\{u_k\} \subset C_c^\infty(\Omega)$ satisfying

$$(14) \quad (a) \ \|u_k\|_p = 1, \quad (b) \ \|\Delta u_k\|_p \rightarrow 0.$$

Note that, by Corollary 3.13, $\{u_k\}$ is bounded in $H_0^{1,p}(\Omega)$. Therefore, the Rellich-Kondrachov compactness theorem yields the existence of a subsequence $\{u_{k'}\}$ that converges strongly in L^p to a function $u \in L^p(\Omega)$. In fact, $u \in C^0(\overline{\Omega})$ if $p > m$. It follows from (14) (a) that

$$(15) \quad \|u\|_p = 1.$$

Now, by (12) and Corollary 3.13, $\{u_{k'}\}$ is bounded in the reflexive Banach space $H_0^{2,p}(\Omega)$. Therefore, a subsequence $\{u_{k''}\}$ converges weakly in $H_0^{2,p}(\Omega)$ and the weak limit is $u \in H_0^{2,p}(\Omega)$. In particular, by Remark 3.3.1, we have that the distributional Laplacian of u is a Borel function $\Delta u \in L^p(\Omega)$ and, furthermore, for every $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned} \int_{\Omega} \varphi \Delta u_{k''} d\mu &= - \int_{\Omega} \text{tr} \circ \text{Hess}(u_{k''}) \cdot \varphi d\mu \\ &= \int_{\Omega} (-\text{Hess}(u_{k''}), \text{tr}^\dagger(\varphi)) d\mu \\ &\rightarrow \int_{\Omega} (-\text{Hess}(u), \text{tr}^\dagger(\varphi)) d\mu \\ &= - \int_{\Omega} \text{tr} \circ \text{Hess}(u) \cdot \varphi d\mu \\ &= \int_{\Omega} \varphi \Delta u d\mu. \end{aligned}$$

On the other hand, by (14) (b),

$$\int_{\Omega} \varphi \Delta u_{k''} d\mu \rightarrow 0,$$

thus proving that $u \in H_0^{2,p}(\Omega)$ is a strong solution of the Laplace equation:

$$\Delta u = 0 \text{ a.e. in } \Omega.$$

By elliptic regularity, Theorem 9.19 in [16], since the Laplace-Beltrami operator is uniformly elliptic with smooth coefficients in $\overline{\Omega}$, we deduce

that $u \in C^\infty(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. The usual maximum principle then implies that $u = 0$. Obviously, this contradicts (15).

b) Let $u \in C_c^\infty(M)$. Take an open subset $\Omega_1 \subset M$ such that $\Omega \subset\subset \Omega_1 \subset\subset M$ and a function $\xi \in C_c^\infty(\Omega_1)$ with $0 \leq \xi \leq 1$, and $\xi = 1$ on Ω , which we are going to use in order to glue two CZ(p)'s together. To this end, set $\phi := (1 - \xi)$. By the validity of CZ(p) on Ω_1 (by part a)) and on $M \setminus \overline{\Omega}$, and writing $u = \xi u + \phi u$, with $\xi u \in C_c^\infty(\Omega_1)$, $\phi u \in C_c^\infty(M \setminus \overline{\Omega})$, we get $A, B > 0$, independent of u , such that

$$\begin{aligned} \|\text{Hess}(u)\|_p &\leq \|\text{Hess}(\xi u)\|_p + \|\text{Hess}(\phi u)\|_p \\ &\leq A \left(\|\Delta(\xi u)\|_p + \|\xi u\|_p \right) + B \left(\|\Delta(\phi u)\|_p + \|\phi u\|_p \right) \\ &\leq A \left(\|u \Delta \xi\|_p + \|\xi \Delta u\|_p + \| |\text{grad}(\xi)| \cdot |\text{grad}(u)| \|_p + \|\xi u\|_p \right) \\ &\quad + B \left(\|u \Delta \phi\|_p + \|\phi \Delta u\|_p + \| |\text{grad}(\phi)| \cdot |\text{grad}(u)| \|_p + \|\phi u\|_p \right) \\ &\leq C \left(\|\Delta u\|_p + \|\text{grad}(u)\|_p + \|u\|_p \right), \end{aligned}$$

with

$$\begin{aligned} 0 < C := & A \max \left(\|\Delta \xi\|_\infty, \|\xi\|_\infty, \|\text{grad}(\xi)\|_\infty \right) \\ & + B \max \left(\|\Delta \phi\|_\infty, \|\phi\|_\infty, \|\text{grad}(\phi)\|_\infty \right). \end{aligned}$$

Interpolating with Proposition 3.12 a), the latter inequality completes the proof. \square

We have seen that CZ(p) always holds on relatively compact domains, however, in general one may have a rough control on the constants. We close this section with the following Theorem 3.16 where we prove a much more precise CZ(p) on sufficiently small geodesic balls. To this end recall the definition of $r_{Q,k,\alpha}(x)$, the $C^{k,\alpha}$ -harmonic radius with accuracy Q at x (cf. Appendix A).

Theorem 3.16. *Fix an arbitrary $x \in M$. Then for all $1 < p < \infty$, all*

$$(16) \quad 0 < r < r_{2,1,1/2}(x)/2,$$

and all real numbers D with

$$r_{2,1,1/2}(B_{r_{2,1,1/2}}(x)) = \inf_{z \in B_{r_{2,1,1/2}}(x)} r_{2,1,1/2}(z) \geq D > 0,$$

there is a constant $C = C(r, p, m, D) > 0$, such that for all $u \in \mathcal{C}_c^\infty(M)$ one has

$$(17) \quad \left\| 1_{B_{r/2}(x)} \text{Hess}(u) \right\|_p \leq C \left(\left\| 1_{B_{2r}(x)} u \right\|_p + \left\| 1_{B_{2r}(x)} \Delta u \right\|_p + \left\| 1_{B_{2r}(x)} \text{grad}(u) \right\|_p \right).$$

In particular, with some $\tilde{C} = \tilde{C}(r, p, m, D) > 0$, for all $u \in \mathcal{C}_c^\infty(B_{r/2}(x))$ one has

$$(18) \quad \left\| \text{Hess}(u) \right\|_p \leq C(\|u\|_p + \|\Delta u\|_p).$$

Proof. Let $u \in \mathcal{C}_c^\infty(M)$, let $r^*(x) := r_{2,1,1/2}(x)$, and pick a $\mathcal{C}^{1,1/2}$ -harmonic coordinate system

$$\phi = (y^1, \dots, y^m) : B_{r^*(x)}(x) \longrightarrow \mathbb{R}^m$$

with accuracy $Q = 2$. Then by the properties (A.1) and (A.2) of ϕ and by Remark A.3 we have the following inequalities on $B_{r^*(x)}(x)$,

$$(19) \quad 2^{-1}(\delta_{ij}) \leq (g_{ij}) \leq 2(\delta_{ij}),$$

$$(20) \quad \max_{i,j \in \{1, \dots, m\}} \{\|g_{ij}\|_\infty, \|g^{ij}\|_\infty\} \leq C_1(m),$$

$$(21) \quad \max_{i,j,l \in \{1, \dots, m\}} \{\|\partial_l g_{ij}\|_\infty, \|\partial_l g^{ij}\|_\infty\} \leq C_2(D, m).$$

Since by (19) and (16) we have

$$\phi(B_{r/2}(x)) \subset B_{r/\sqrt{2}}^{\text{eucl}}(0) \subset B_{\sqrt{2}r}^{\text{eucl}}(0) \subset \phi(B_{2r}(x)) \subset \phi(B_{r^*(x)}(x)),$$

applying Theorem 9.11 from [16] with $L = \Delta$ and with the Euclidean balls $\Omega := B_{\sqrt{2}r}^{\text{eucl}}(0)$, $\Omega' := B_{r/\sqrt{2}}^{\text{eucl}}(0)$ implies the existence of a $C_3 = C_3(p, r, m, D) > 0$ such that

$$(22) \quad \int_{B_{r/2}(x)} \sum_{i,j} |\partial_i \partial_j u(y)|^p dy \leq C_3 \int_{B_{2r}(x)} |\Delta u(y)|^p dy + C_3 \int_{B_{2r}(x)} |u(y)|^p dy.$$

One can deduce from (19)-(21) the following pointwise estimate in $B_{r^*(x)}(x)$,

$$(23) \quad |\text{Hess}(u)|^p \leq C_8 \sum_{i,j} |\partial_i \partial_j u|^p + C_5 |\text{grad}(u)|^p,$$

with some $C_8 = C_8(r, m, D, p) > 0$. Indeed, let $\|A\|_{\text{HS}}^2 = \sum_{i,j} A_{ij}^2$ denote the Hilbert-Schmidt norm of a real-valued matrix $A = (A_{ij})$.

Then with $H := (\text{Hess}(u)_{ij})$, $h := (\partial_i \partial_j u)$, $\Gamma := (-\sum_l \Gamma_{ij}^l \partial_l u)$, $G := (g_{ij})$ one has $H = h + \Gamma$ and

$$\begin{aligned} |\text{Hess}(u)| &= \|G^{-1}H\|_{\text{HS}} = \|G^{-1}h + G^{-1}\Gamma\|_{\text{HS}} \\ &\leq \|G^{-1}\|_{\text{HS}} (\|h\|_{\text{HS}} + \|\Gamma\|_{\text{HS}}) \leq C_4 (\|h\|_{\text{HS}} + \|\Gamma\|_{\text{HS}}) \quad \text{in } B_{r^*(x)}(x) \end{aligned}$$

and

$$\|\Gamma\|_{\text{HS}}^2 \leq \sum_{ij} \left(\sum_l \Gamma_{ij}^l \right)^2 \cdot \sum_l (\partial_l u)^2 \leq C_5 |\text{grad}(u)|^2 \quad \text{in } B_{r^*(x)}(x),$$

for some $C_4, C_5 > 0$ depending only on D and m . Whence, we get the estimate on $B_{r^*(x)}(x)$

$$\begin{aligned} |\text{Hess}(u)| &\leq C_6 \sqrt{\sum_{i,j} |\partial_i \partial_j u|^2} + C_6 |\text{grad}(u)|, \\ |\text{Hess}(u)|^p &\leq C_7 \sum_{i,j} |\partial_i \partial_j u|^p + C_7 |\text{grad}(u)|^p \end{aligned}$$

for some $C_6, C_7 > 0$ depending only on D , m and p . This proves the validity of (23). Using this latter in combination with (19) and (22) gives us a $C_9 = C_9(r, m, D, p) > 0$ such that

$$\begin{aligned} &\left\| 1_{B_{r/2}(x)} \text{Hess}(u) \right\|_p \\ &\leq C_9 \left(\left\| 1_{B_{2r}(x)} u \right\|_p + \left\| 1_{B_{2r}(x)} \Delta u \right\|_p + \left\| 1_{B_{2r}(x)} \text{grad}(u) \right\|_p \right). \end{aligned}$$

Finally, (18) follows from by interpolation using Proposition 3.12 a). \square

Let us remark here that an essential point of the estimate from Theorem 3.16 is that C depends on x only through a lower bound D on the local harmonic radius, a fact which makes it possibly to use this result in order to derive $\text{CZ}(p)$ on a large class of noncompact Riemannian manifolds (cf. the proof of Theorem 4.3).

4. GEOMETRIC CRITERIA FOR GLOBAL CALDERÓN-ZYGMUND INEQUALITIES

This section is devoted to Riemann geometric criteria for the validity of global $\text{CZ}(p)$ inequalities.

Let us start with the $p = 2$ case: Here, in view of Bochner's equality, it is easy to give a rather complete answer: $\text{CZ}(2)$ always holds globally in a strong, “infinitesimal” way, under a global lower bound on the Ricci curvature, and furthermore this result does not even require geodesic completeness:

Proposition 4.1. *Assume that $\text{Ric} \geq -C^2$ for some constant $C \in \mathbb{R}$, meaning as usual that*

$$\text{Ric}(X, X) \geq -C^2|X|^2 \text{ for all vector fields } X \in \Gamma_{\mathbb{C}^\infty}(M, TM).$$

Then CZ(2) holds in the following “infinitesimal” way: For every $\varepsilon > 0$ and every $u \in \mathbb{C}_c^\infty(M)$ one has

$$\|\text{Hess}(u)\|_2^2 \leq \frac{C\varepsilon^2}{2} \|u\|_2^2 + \left(1 + \frac{C^2}{2\varepsilon^2}\right) \|\Delta u\|_2^2.$$

Proof. By Bochner’s equality we have

$$\begin{aligned} & |\text{Hess}(u)|^2 \\ &= -\frac{1}{2}\Delta|\text{grad}(u)|^2 + (\text{grad}(u), \text{grad}(\Delta u)) - \text{Ric}(\text{grad}(u), \text{grad}(u)) \\ &= -\frac{1}{2}\Delta|\text{grad}(u)|^2 + (du, d\Delta u) - \text{Ric}(\text{grad}(u), \text{grad}(u)). \end{aligned}$$

Now the claim follows easily from integrating this identity, using integration by parts, $\Delta u = d^\dagger du$ and the inequality

$$ab \leq \frac{a^2}{2\varepsilon^2} + \frac{\varepsilon^2 b^2}{2},$$

valid for $a, b \geq 0$. □

On the other hand, it is necessary for CZ(2) to have *some* control on the curvature, as can be seen from:

Theorem 4.2. *There exists a 2-dimensional, geodesically complete Riemannian manifold N with unbounded Gaussian curvature and such that CZ(2) fails on N .*

The proof of Theorem 4.2 is given in Section 5.

For arbitrary values of p , the situation is much more complicated. Here, we found the following two criteria, which can also be considered as the main result of this paper.

The first result covers the whole L^p -scale in a great generality:

Theorem 4.3. *Let $1 < p < \infty$ and assume $\|\text{Ric}\|_\infty < \infty$, $r_{\text{inj}}(M) > 0$. Then there is a*

$$C = C(m, p, \|\text{Ric}\|_\infty, r_{\text{inj}}(M)) > 0,$$

such that for all $u \in \mathbb{C}_c^\infty(M)$ one has

$$(24) \quad \|\text{Hess}(u)\|_p \leq C(\|u\|_p + \|\Delta u\|_p).$$

Remark 4.4. *Note that, using the usual definition of geodesic completeness in terms of the exponential function, it is elementary to see that a positive injectivity radius automatically implies geodesically completeness.*

The second result is concerned with the $1 < p \leq 2$ case in a slightly different setting: the geometry of the manifold is bounded up to order one but the injectivity radius condition is replaced with a kind of generalized volume doubling assumption. The proof of this result is of independent interest because it points out a deep relation between Calderón-Zygmund inequalities and covariant Riesz transforms:

Theorem 4.5. *Let $1 < p \leq 2$. Assume that M is geodesically complete with $\|R\|_\infty < \infty$, $\|\nabla R\|_\infty < \infty$, and that there are constants $D \geq 1$, $0 \leq \delta < 2$ with*

$$(25) \quad \mu(B_{tr}(x)) \leq Dt^D e^{t^\delta + r^\delta} \mu(B_r(x)) \text{ for all } x \in M, r > 0, t \geq 1.$$

Then there is a

$$C = C(m, p, \|R\|_\infty, \|\nabla R\|_\infty, D, \delta) > 0,$$

such that for all $u \in C_c^\infty(M)$ one has

$$(26) \quad \|\text{Hess}(u)\|_p \leq C(\|u\|_p + \|\Delta u\|_p).$$

Remark 4.6. *If M is geodesically complete with $\text{Ric} \geq 0$, then one has the doubling condition*

$$\mu(B_{2r}(x)) \leq 2^m \mu(B_r(x)) \text{ for all } r > 0, x \in M,$$

which easily implies

$$\mu(B_{tr}(x)) \leq 2^{2m} t^m \mu(B_r(x)) \text{ for all } r > 0, t \geq 1, x \in M,$$

so that (25) is satisfied in this situation (with constants that only depend on m).

On the other hand, nonnegative Ricci curvature is not necessary for (25):

Example 4.7. *Let (N, h) be a compact Riemannian manifold of dimension $m - 1$ and let $(M, g) = (N \times \mathbb{R}, h + dt \otimes dt)$. Then, (M, g) satisfies the assumptions of Theorem 4.5 (actually, also those of Theorem 4.3). Indeed, M is co-compact, hence it has bounded geometry up to order ∞ . On the other hand, there exists a constant $C > 0$ depending on the geometry of N such that, for every $(p_0, t_0) \in M$, $R > 0$ and $t \geq 1$*

$$\frac{\mu_M(B_{tR}^M((p_0, t_0)))}{\mu_M(B_R^M((p_0, t_0)))} \leq Ct^m.$$

Indeed, since

$$\max(d_N, d_{\mathbb{R}}) \leq d_M = \sqrt{d_N^2 + d_{\mathbb{R}}^2} \leq \sqrt{2} \max(d_N, d_{\mathbb{R}})$$

we have

$$B_{R/\sqrt{2}}^N(p_0) \times B_{R/\sqrt{2}}^{\mathbb{R}}(t_0) \subseteq B_R^M((p_0, t_0)) \subseteq B_R^N(p_0) \times B_R^{\mathbb{R}}(t_0)$$

proving that

$$\frac{\mu_M(B_{tR}^M((p_0, t_0)))}{\mu_M(B_R^M((p_0, t_0)))} \leq \sqrt{2}t \cdot \frac{\mu_N(B_{tR}^N(p_0))}{\mu_N(B_{R/\sqrt{2}}^N(p_0))}.$$

Therefore, we are reduced to show that

$$\frac{\mu_N(B_{tR}^N(p_0))}{\mu_N(B_{R/\sqrt{2}}^N(p_0))} \leq Ct^{m-1}.$$

To this end, note that, if

$$tR \leq \frac{r_{\text{inj}}(N)}{2}.$$

then, the desired inequality follows from volume comparison. Indeed, let

$$-K^2 \leq \text{Sec}_N \leq K^2,$$

then the continuous functions $\alpha_1, \alpha_2 : [0, r_{\text{inj}}(N)/2] \rightarrow (0, \infty)$ defined by (P. Petersen notation, [28])

$$\begin{aligned} \alpha_1(r) &= \frac{\int_0^r \text{sn}_{K^2}^{m-2}(s) \, ds}{r^{m-1}} \\ \alpha_2(r) &= \frac{\int_0^r \text{sn}_{-K^2}^{m-2}(s) \, ds}{r^{m-1}} \end{aligned}$$

satisfy

$$A_i \leq \alpha_i(r) \leq B_i$$

where $A_i, B_i > 0$ are constants depending only on K, m and $r_{\text{inj}}(N)$. It follows that

$$\frac{\mu_N(B_{tR}^N(p_0))}{\mu_N(B_{R/\sqrt{2}}^N(p_0))} \leq \frac{B_2}{A_1} \frac{(tR)^{m-1}}{(R/\sqrt{2})^{m-1}} = Ct^{m-1},$$

as claimed. On the other hand, if

$$tR > \frac{r_{\text{inj}}(N)}{2},$$

since

$$\frac{\mu_N(B_{tR}^N(p_0))}{\mu_N(B_{R/\sqrt{2}}^N(p_0))} \leq \frac{\mu_N(N)}{\mu_N(B_{r_{\text{inj}}(N)/(t2\sqrt{2})}^N(p_0))}$$

and

$$\frac{r_{\text{inj}}(N)}{t2\sqrt{2}} \leq \frac{r_{\text{inj}}(N)}{2}$$

using again volume comparison we get

$$\mu_N(B_{r_{\text{inj}}(N)/(t2\sqrt{2})}^N(p_0)) \geq A_1 \left(\frac{r_{\text{inj}}(N)}{t2\sqrt{2}} \right)^{m-1}$$

and, hence,

$$\frac{\mu_N(B_{tR}^N(p_0))}{\mu_N(B_{R/\sqrt{2}}^N(p_0))} \leq \frac{\mu_N(N)}{A_1} \left(\frac{2\sqrt{2}}{r_{\text{inj}}(N)} \right)^{m-1} \cdot t^{m-1}.$$

This completes the proof.

The rest of this section is devoted to the proof of Theorem 4.3 and of Theorem 4.5, respectively.

We will need the following auxiliary result (see for example Lemma 1.6 in [21] and its proof) for the former:

Lemma 4.8. *Assume that M is geodesically complete with $\text{Ric} \geq -C$ for some $C > 0$. Then for any $r > 0$ there exists a sequence of points $\{x_i\} \subset M$ and a natural number $N = N(m, r, C) < \infty$, such that*

- $B_{r/4}(x_i) \cap B_{r/4}(x_j) = \emptyset$ for all $i, j \in \mathbb{N}$ with $i \neq j$,
- $\bigcup_{i \in \mathbb{N}} B_{r/2}(x_i) = M$,
- the intersection multiplicity of the system $\{B_{2r}(x_i) | i \in \mathbb{N}\}$ is $\leq N$.

Now we can give the

Proof of Theorem 4.3. By Theorem A.4 there is a

$$D = D(m, r_{\text{inj}}(M), \|\text{Ric}\|_\infty) > 0$$

such that $r_{2,1,1/2}(M) \geq D$. Let $r := D/2$. We take a covering $\bigcup_{i \in \mathbb{N}} B_{r/2}(x_i) = M$ as in Lemma 4.8. By Theorem 3.16 we have a $c = c(r, p, m, D) > 0$ such that, for all $i \in \mathbb{N}$, all $u \in C_c^\infty(M)$,

$$\begin{aligned} & \int_{B_{r/2}(x_i)} |\text{Hess}(u)|^p d\mu \\ & \leq c \int_{B_{2r}(x_i)} |\Delta u|^p d\mu + c \int_{B_{2r}(x_i)} |\text{grad}(u)|^p d\mu + c \int_{B_{2r}(x_i)} |u|^p d\mu, \end{aligned}$$

so summing over i and using monotone convergence we get

$$\begin{aligned} \int_M |\text{Hess}(u)|^p d\mu &\leq \sum_i \int_{B_{r/2}(x_i)} |\text{Hess}(u)|^p d\mu \\ &\leq c \int_M \sum_i 1_{B_{2r}(x_i)} |\Delta u|^p d\mu + c \int_M \sum_i 1_{B_{2r}(x_i)} |\text{grad}(u)|^p d\mu \\ &\quad + c \int_M \sum_i 1_{B_{2r}(x_i)} |u|^p d\mu, \end{aligned}$$

which by Lemma 4.8 gives

$$\|\text{Hess}(u)\|_p \leq (cN)^{1/p} \left(\|\Delta u\|_p + \|\text{grad}(u)\|_p + \|u\|_p \right).$$

A use of Proposition 3.12 a) completes the proof. \square

Remark 4.9. *Obviously, a similar argument can be used to prove Theorem 3.14 a). Simply cover the compact domain $\bar{\Omega}$ with a finite number of balls $B_{r/2}$ with $0 < 2r < r_{2,1,1/2}(\bar{\Omega})$.*

Finally, we give the proof of Theorem 4.5, which as we have already remarked in the introduction, uses the machinery of covariant Riesz-transforms. We will need the following auxiliary Hilbert space lemma:

Lemma 4.10. *Let S be a densely defined closed linear operator from a Hilbert space \mathcal{H}_1 to a Hilbert space \mathcal{H}_2 , and let T be a bounded self-adjoint operator in \mathcal{H}_1 . Then for any $\lambda > 0$ with $T \geq -\lambda$ one has*

$$\|S(S^*S + T + \lambda + 1)^{-1/2}\| \leq 1.$$

Proof. Firstly, the polar decomposition of S reads $S = U(S^*S)^{1/2}$, with a partial isometry U from \mathcal{H}_1 to \mathcal{H}_2 whose domain of isometry contains the range of $(S^*S)^{1/2}$. Secondly, we have

$$S^*S + T + \lambda + 1 \geq S^*S + 1,$$

which in this case means nothing but

$$\|(S^*S + T + \lambda + 1)^{1/2}f\| \geq \|(S^*S + 1)^{1/2}f\|$$

for all f in the domain of definition of $(S^*S)^{1/2}$, in particular,

$$\|(S^*S + 1)^{1/2}(S^*S + T + \lambda + 1)^{-1/2}h\| \leq \|h\| \text{ for all } h \in \mathcal{H}_2$$

so

$$\|(S^*S + 1)^{1/2}(S^*S + T + \lambda + 1)^{-1/2}\| \leq 1.$$

Now we can estimate as follows

$$\begin{aligned}
& \left\| S(S^*S + T + \lambda + 1)^{-\frac{1}{2}} \right\| \\
&= \left\| S(S^*S + 1)^{-\frac{1}{2}}(S^*S + 1)^{\frac{1}{2}}(S^*S + T + \lambda + 1)^{-\frac{1}{2}} \right\| \\
&\leq \left\| (S^*S)^{\frac{1}{2}}(S^*S + 1)^{-\frac{1}{2}}(S^*S + 1)^{\frac{1}{2}}(S^*S + T + \lambda + 1)^{-\frac{1}{2}} \right\| \\
&\leq \left\| (S^*S)^{\frac{1}{2}}(S^*S + 1)^{-\frac{1}{2}} \right\| \leq \sup_{t \geq 0} \sqrt{t}/\sqrt{t+1} \leq 1,
\end{aligned}$$

where we have used the spectral calculus for the last norm bound. \square

Proof of Theorem 4.5. The assumption $\|\mathbf{R}\|_\infty < \infty$ implies

$$(27) \quad -c \leq \text{Ric} \leq \tilde{c} \text{ for some } c = c(\|\mathbf{R}\|_\infty, m) > 0, \tilde{c} = \tilde{c}(\|\mathbf{R}\|_\infty, m) > 0,$$

and we set $\sigma := c + 1 > 0$. We are going to prove the existence of a

$$C = C(m, p, \|\mathbf{R}\|_\infty, \|\nabla \mathbf{R}\|_\infty, D, \delta) > 0$$

such that for all $u \in \mathbf{C}_c^\infty(M)$ one has

$$(28) \quad \left\| \nabla d(\Delta_0 + \sigma)^{-1} u \right\|_p \leq C \|u\|_p,$$

which under geodesic completeness is equivalent to

$$(29) \quad \left\| \nabla(\Delta_1 + \sigma)^{-1/2} d(\Delta_0 + \sigma)^{-1/2} u \right\|_p \leq C \|u\|_p.$$

To this end, we start by observing that by a classical result on Riesz-transforms of functions by Bakry [2] (see also [24, 25] for the weighted case), there is a constant $C_1 = C_1(p) > 0$ with

$$(30) \quad \left\| d(\Delta_0 + \sigma)^{-1/2} \right\|_{p,p} \leq C_1.$$

Next, we are going to use Theorem 4.1 in [30] in combination with Example 2.6 therein to treat the $\nabla(\Delta_1 + \sigma)^{-1/2}$ part: To this end, let us first note that

$$\Delta_1 = \nabla^\dagger \nabla + \text{Ric}(\sharp, \sharp),$$

so that applying Lemma 4.10 (where we omit obvious essential self-adjointness arguments) with $S = \nabla$ (on 1-forms), and $T = \text{Ric}(\sharp, \sharp)$, which is read as a self-adjoint multiplication operator, bounded by assumption (27), we get that the operator

$$T_\sigma := \nabla(\Delta_1 + \sigma)^{-1/2}$$

from Theorem 4.1 in [30] is bounded in the L^2 -sense, with operator norm ≤ 1 . It remains to check the corresponding assumptions A and B from [30]: Here, the validity of assumption A follows immediately from our curvature assumptions and (27), cf. Example 2.6 from [30].

Assumption *B1* follows from the Laplacian comparison theorem and (27), and assumption *B3* is implied by the usual Li-Yau heat kernel estimates, using again (27). Finally, *B2* is precisely our volume assumption (25). Thus, by Theorem 4.1 in [30] we get a

$$C_2 = C_2(m, p, \|R\|_\infty, \|\nabla R\|_\infty, D, \delta) > 0$$

with

$$\|\nabla(\Delta_1 + \sigma)^{-1/2}\|_{p,p} \leq C_2,$$

which, in combination with (30), proves (29) with $C := C_1 C_2$, thus (28), and the proof is complete. \square

5. PROOF OF THEOREM 4.2

In this section, we construct an explicit example of a complete Riemannian manifold M with unbounded curvature and that does not support the global L^2 -Calderón-Zygmund inequality

$$\text{CZ}(2) \quad \|\text{Hess}(u)\|_2 \leq C(\|\Delta u\|_2 + \|u\|_2), \quad u \in C_c^\infty(M).$$

Roughly speaking, in order to violate CZ(2), the idea is to minimize the contribution of Δu with respect to $\text{Hess}(u)$. Clearly, the best way to do this would be to choose u harmonic (and not affine) but this is impossible because u has compact support. To overcome the problem, we can take u as the composition of a proper harmonic function with a singularity at the origin and a cut-off function of \mathbb{R} , compactly supported in $(0, \infty)$. Using this composition we get rid of the singularity and produce a smooth, compactly supported function whose L^2 -norm of the Laplacian can be small when compared with that of the Hessian. We shall implement this construction on a model manifold where, for rotationally symmetric functions, the expressions of the L^2 -norms involved in CZ(2) are very explicit and directly related to the geometry of the underlying space.

By an m -dimensional model manifold \mathbb{R}_σ^m we mean the Euclidean space \mathbb{R}^m endowed with the smooth, complete Riemannian metric that, in polar coordinates, writes as

$$g = dr \otimes dr + \sigma^2(r) g_{\mathbb{S}^{m-1}},$$

where $g_{\mathbb{S}^{m-1}}$ is the standard metric of \mathbb{S}^{m-1} and $\sigma : [0, \infty) \rightarrow [0, \infty)$ is a smooth function satisfying the following structural conditions:

- (a) $\sigma^{(2k)}(0) = 0, \forall k = 0, 1, \dots$
- (b) $\sigma'(0) = 1$
- (c) $\sigma(t) > 0, \forall t > 0$.

We can always identify σ with its smooth, odd extension $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ such that $\sigma(t) = -\sigma(-t)$ for every $t \leq 0$. Recall that the sectional curvatures of \mathbb{R}_σ^m are given by

$$\begin{aligned} \text{Sec}(X \wedge \nabla r) &= -\frac{\sigma''}{\sigma} \\ \text{Sec}(X \wedge Y) &= \frac{1 - (\sigma')^2}{\sigma^2}, \end{aligned}$$

for every g -orthonormal vectors $X, Y \in \nabla r^\perp$, where ∇r represents the radial direction. Moreover, observe that the Riemannian measure of \mathbb{R}_σ^m is given by

$$d\mu = \sigma^{m-1}(r) \cdot dr \cdot d\mu_{\mathbb{S}^{m-1}},$$

where $d\mu_{\mathbb{S}^{m-1}}$ denotes the canonical Riemannian measure on \mathbb{S}^{m-1} .

Let us assume that

$$\int_1^\infty \frac{dt}{\sigma^{m-1}(t)} = \infty.$$

From the potential theoretic viewpoint, this means that \mathbb{R}_σ^m is parabolic, namely, the minimal positive Green kernel of the Laplace-Beltrami operator of \mathbb{R}_σ^m is identically ∞ . Then,

$$G(r) = \int_1^r \frac{dt}{\sigma^{m-1}(t)}$$

is a smooth, positive, strictly-increasing function on $(0, \infty)$ satisfying

$$G(r) \begin{cases} = \infty & \text{if } r = \infty \\ > 0 & \text{if } r > 1 \\ = 0 & \text{if } r = 1 \\ < 0 & \text{if } 0 < r < 1 \\ = -\infty & \text{if } r = 0^+. \end{cases}$$

Moreover, $G(r)$ gives rise to a smooth, rotationally symmetric harmonic function $G(x)$ on $\mathbb{R}_\sigma^m \setminus \{0\}$. In particular:

$$\Delta G = G'' + (m-1) \frac{\sigma'}{\sigma} G' = 0, \text{ on } \mathbb{R}_\sigma^m \setminus \{0\}.$$

We need the following computational Lemma.

Lemma 5.1. *Let \mathbb{R}_σ^m be a complete, parabolic, model manifold so that $\sigma^{1-m} \notin L^1(+\infty)$. Let, as above,*

$$G(r) = \int_1^r \frac{1}{\sigma^{m-1}(t)} dt,$$

and let $\{\alpha_k\}, \{\beta_k\} \subset (0, \infty)$ be two sequences such that $1 < \alpha_k < \beta_k$. Assume further that for any k one has given a function $\phi_k \in C_c^\infty(0, \infty)$

which satisfies $\text{supp}(\phi_k) \subset [\alpha_k, \beta_k]$, and define $u_k \in \mathcal{C}^\infty(\mathbb{R}_\sigma^m)$ by setting $u_k(x) = \phi_k(G(x))$. Then, each u_k is in fact compactly supported in

$$\{\alpha_k \leq G \leq \beta_k\} \subset \mathbb{R}_\sigma^m,$$

and one has

$$\|\text{Hess}(u_k)\|_2^2 \geq \omega_m \int_{\alpha_k}^{\beta_k} (\phi'_k(s))^2 \left(\frac{\sigma'}{\sigma}(G^{-1}(s)) \right)^2 ds,$$

$$\|\Delta u_k\|_2^2 = \omega_m \int_{\alpha_k}^{\beta_k} \frac{(\phi''_k(s))^2}{(\sigma(G^{-1}(s)))^{2(m-1)}} ds,$$

$$\|u_k\|_2^2 = \omega_m \int_{\alpha_k}^{\beta_k} (\phi_k(s))^2 (\sigma(G^{-1}(s)))^{2(m-1)} ds,$$

where $\omega_m > 0$ is a dimensional constant.

Proof. Recall that

$$\text{Hess}(u_k) = u''_k \cdot dr \otimes dr + \frac{\sigma'}{\sigma} u'_k \cdot \sigma^2 g_{\mathbb{S}^{m-1}}.$$

Therefore, we have

$$\begin{aligned} |\text{Hess}(u_k)|^2 &= (u''_k)^2 + (m-1)(u'_k)^2 \left(\frac{\sigma'}{\sigma} \right)^2 \\ &\geq (u'_k)^2 \left(\frac{\sigma'}{\sigma} \right)^2. \end{aligned}$$

Since

$$\begin{aligned} u'_k(r) &= \phi'_k(G) G' \\ &= \phi'_k(G) \frac{1}{\sigma^{m-1}} \end{aligned}$$

we get

$$\begin{aligned} |\text{Hess}(u_k)|^2 &\geq (u'_k)^2 \left(\frac{\sigma'}{\sigma} \right)^2 \\ &= (\phi'_k(G))^2 (G')^2 \left(\frac{\sigma'}{\sigma} \right)^2 \\ &= (\phi'_k(G))^2 \left(\frac{\sigma'}{\sigma} \right)^2 \frac{G'}{\sigma^{m-1}}. \end{aligned}$$

In particular, letting ω_m be the volume of the standard $(m-1)$ -sphere,

$$\begin{aligned}
\|\text{Hess}(u_k)\|_2^2 &= \omega_m \int_0^\infty |\text{Hess}(u_k)|^2 \sigma^{m-1} dt \\
&\geq \omega_m \int_0^\infty (\phi'_k(G))^2 \left(\frac{\sigma'}{\sigma}\right)^2 G' dt \\
&= \omega_m \int_{G^{-1}(\alpha_k)}^{G^{-1}(\beta_k)} (\phi'_k(G))^2 \left(\frac{\sigma'}{\sigma}\right)^2 G' dt \\
&= \omega_m \int_{\alpha_k}^{\beta_k} (\phi'_k(s))^2 \left(\frac{\sigma'}{\sigma}(G^{-1})\right)^2 ds
\end{aligned}$$

where, in the last equality, we have used the change of variable $G(t) = s$. Similarly, on noting that

$$\begin{aligned}
u_k''(r) &= \phi_k''(G) (G')^2 + \phi_k'(G) G'' \\
&= \phi_k''(G) \frac{1}{\sigma^{2(m-1)}} + \phi_k'(G) G'',
\end{aligned}$$

using also the harmonicity of G , we compute

$$\begin{aligned}
\Delta u_k &= u_k'' + (m-1) \frac{\sigma'}{\sigma} u_k' \\
&= \phi_k''(G) \frac{1}{\sigma^{2(m-1)}} + \phi_k'(G) \left(G'' + (m-1) \frac{\sigma'}{\sigma} G' \right) \\
&= \phi_k''(G) \frac{1}{\sigma^{2(m-1)}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\Delta u_k\|_2^2 &= \omega_m \int_0^\infty (\phi_k''(G))^2 \frac{1}{\sigma^{3(m-1)}} dt \\
&= \omega_m \int_{G^{-1}(\alpha_k)}^{G^{-1}(\beta_k)} (\phi_k''(G))^2 \frac{1}{\sigma^{2(m-1)}} G' dt \\
&= \omega_m \int_{\alpha_k}^{\beta_k} (\phi_k''(s))^2 \frac{1}{(\sigma(G^{-1}))^{2(m-1)}} ds
\end{aligned}$$

Finally, we compute

$$\begin{aligned}\|u_k\|_2^2 &= \omega_m \int_0^\infty (\phi_k(G))^2 \sigma^{m-1} dt \\ &= \omega_m \int_{G^{-1}(\alpha_k)}^{G^{-1}(\beta_k)} (\phi_k(G))^2 \sigma^{2(m-1)} G' dt \\ &= \omega_m \int_{\alpha_k}^{\beta_k} (\phi_k(s))^2 (\sigma(G^{-1}))^{2(m-1)} ds.\end{aligned}$$

This completes the proof. \square

Now we proceed with the choice of the warping function σ and of the cut-off functions ϕ_k in such a way that CZ(2) is violated along the corresponding sequence of test-functions u_k . To this end, we begin by taking

$$m = 2, \alpha_k = k, \beta_k = k + 1.$$

Next, we choose $\sigma(t)$ in such a way that

$$t \leq \sigma(t) \leq t + 1, \quad t > 1.$$

Remark 5.2. We explicitly note that, by definition of G ,

$$\log\left(\frac{t+1}{2}\right) \leq G(t) \leq \log(t), \quad t > 1.$$

It follows that

$$e^s \leq G^{-1}(s) \leq 2e^s - 1, \quad s > 0.$$

In particular,

$$e^k \leq G^{-1}(s) \leq 2e^{k+1} - 1, \quad \text{on } [k, k+1].$$

Whence, since

$$G^{-1}(k+1) - G^{-1}(k) \geq e^{k+1} - 2e^k + 1 = e^k(e-2) + 1 > 1$$

we also deduce that, for each k , there exists some integer $h = h(k) > k$ such that

$$[h, h+1] \subseteq [G^{-1}(k), G^{-1}(k+1)].$$

Furthermore,

$$e^s \leq \sigma(G^{-1}(s)) \leq 2e^s.$$

These estimates will be used repeatedly in the sequel.

We also require that $\sigma(t)$ oscillates in each interval $[k, k+1]$ with a slope that increases with k . We can model the oscillating part by segments like

$$t \mapsto \bar{t} + (t - \bar{t})(\varepsilon_k + 1)/\varepsilon_k \quad \text{on } [\bar{t}, \bar{t} + \varepsilon_k],$$

and

$$t \mapsto \bar{t} + 2\varepsilon_k + (\bar{t} + 2\varepsilon_k - t) / \varepsilon_k \text{ on } [\bar{t} + \varepsilon_k, \bar{t} + 2\varepsilon_k],$$

with $\varepsilon_k \rightarrow 0+$. The upper and lower angles are smoothened out in regions as close to the vertices as we desire. The smoothing can be realized via concave (resp. convex) functions; see [15].

Remark 5.3. *By construction, each rectilinear portion of $(\sigma')^2$ grows like $1/\varepsilon_k^2$ on an interval of approximate length ε_k .*

Remark 5.4. *The smoothing is obtained via functions of increasingly high second derivative. It follows that the Gaussian curvature of \mathbb{R}_σ^2 explodes to ∞ as the oscillatory part becomes closer and closer to vertical segments. We also point out that, in dimensions $m \geq 3$ this construction gives rise to a model manifold whose sectional curvatures, both radial and tangential, explode to ∞ . Finally, observe that, due to the profile of σ , the manifold should have vanishing injectivity radius (although, at the pole $0 \in \mathbb{R}_\sigma^2$, it holds that $r_{\text{inj}}(0) = \infty$).*

To conclude, we choose $\phi_k(t) = \phi(t - k)$ where $\phi(t)$ is compactly supported in $[0, 1]$ and satisfies $\phi(t) = 2t$ on $[1/4, 1/2]$. In this way, $\phi'_k \equiv 2$ on the interval of fixed length $[k + 1/4, k + 1/2]$ so to capture some of the oscillations of σ . Note also that $\|\phi_k\|_\infty$ and $\|\phi'_k\|_\infty$ are uniformly bounded.

Now, according to Lemma 5.1, we have the following estimates:

$$\|u_k\|_2^2 = \omega_2 \int_k^{k+1} (\phi_k(s))^2 (\sigma(G^{-1}(s)))^2 ds \leq Ce^{2k},$$

and

$$\begin{aligned} \|\Delta u_k\|_2^2 &= \omega_2 \int_k^{k+1} \frac{(\phi'_k(s))^2}{(\sigma(G^{-1}(s)))^2} ds \\ &\leq C \int_k^{k+1} \frac{ds}{e^{2s}} = \frac{C}{e^{2k}} \end{aligned}$$

and, finally,

$$\begin{aligned}
 \|\text{Hess}(u_k)\|_2^2 &\geq \omega_2 \int_k^{k+1} (\phi'_k(s))^2 \left(\frac{\sigma'}{\sigma}(G^{-1}(s)) \right)^2 ds \\
 &\geq \frac{\omega_2}{e^{2(k+1)}} \int_k^{k+1} (\phi'_k(s))^2 (\sigma'(G^{-1}(s)))^2 ds \\
 &\geq \frac{C}{e^{2(k+1)}} \int_0^{\varepsilon_h} \frac{1}{\varepsilon_h^2} ds, \quad h = h(k) > k, \\
 &\geq \frac{C}{e^{2k}} \frac{1}{\varepsilon_k}.
 \end{aligned}$$

Whence, we deduce that we can choose $\varepsilon_k \searrow 0$ in such a way that CZ(2) is violated. This completes the proof of Theorem 4.2.

6. CALDERÓN-ZYGMUND INEQUALITIES ON H -HYPERSURFACES

Let $\mathbb{M}^{m+1}(c)$ denote the complete, simply connected space-form of constant sectional curvature $c \leq 0$. In this section we explore the validity of the L^p -Calderón-Zygmund inequalities on a largely investigated class of submanifolds of $\mathbb{M}^{m+1}(c)$: the hypersurfaces of constant mean curvature $H \in \mathbb{R}$ (H -hypersurfaces for short) with finite total scalar curvature.

Let $f : M \rightarrow \mathbb{M}^{m+1}(c)$ be a complete, connected, oriented, isometrically immersed submanifold of dimension $\dim M = m \geq 3$. Its second fundamental tensor, with respect to a chosen Gauss map ν , is denoted by \mathbf{II} . The corresponding mean curvature vector field is $\mathbf{H} = \text{trace}(\mathbf{II})/m$. We write $\mathbf{H} = H\nu$, where the smooth function H is the mean curvature function of the hypersurface, and we assume that H is constant. The *total curvature* of the constant mean curvature hypersurface M is the L^m -norm of its traceless second fundamental tensor $\Phi = \mathbf{II} - \mathbf{H}g$. We say that M has *finite total curvature* if $\|\Phi\|_m < \infty$. In case $\mathbf{H} = 0$ the hypersurface is called *minimal* and the finite total curvature condition reduces to $\|\mathbf{II}\|_m < \infty$.

A complete, oriented H -hypersurface M in $\mathbb{M}^{m+1}(c)$ of finite total curvature must be necessarily closed provided $H^2 + c > 0$. Indeed, according to [1, 4], the traceless tensor Φ satisfies the decay condition

$$(31) \quad \sup_{M \setminus B_R^M} |\Phi| \rightarrow 0 \text{ as } R \rightarrow \infty.$$

See also [29]. Therefore, by Gauss equations, the Ricci curvature of M is positively pinched outside a compact set, [23], and the compactness conclusion follows from the Bonnet-Myers type theorems in [13].

Since, on the one hand, the condition $H^2 + c > 0$ implies obvious non-existence results and, on the other hand, we are mainly interested in non-compact situations, from now on we assume that

$$(32) \quad H^2 + c \leq 0.$$

In particular, if $c = 0$, then M is minimal.

Under the compatibility condition (32), it is a well known consequence of the curvature estimate (31) that the H -hypersurface is properly immersed and it has a finite number of ends, each of which is diffeomorphic to a cylinder over some compact hypersurface; [1, 6]. Moreover, up to imposing a more stringent pinching on H when $c < 0$, the volume of each end is subjected to a certain growth; [1, 29]. Actually, it is known that *any* complete Riemannian manifold isometrically immersed with *bounded mean curvature* into a Cartan-Hadamard manifold satisfies the *non-collapsing* condition at infinity

$$(33) \quad \inf_{x \in M} \mu(B_1(x)) = v > 0.$$

Indeed, according to [20], such a submanifold enjoys the L^1 -Sobolev inequality

$$\|u\|_{\frac{m}{m-1}} \leq C(\|\text{grad}(u)\|_1 + \|u\|_1),$$

for every $u \in C_c^\infty(M)$ and for some constant $C > 0$ depending on m and $\|\mathbf{H}\|_\infty < +\infty$. Whence, it is standard to deduce the validity of (33) by integrating the differential inequality

$$\mu(B_r(x))^{\frac{m-1}{m}} \leq C \left(\frac{d}{dr} \mu(B_r(x)) + \mu(B_r(x)) \right)$$

that arises from a suitable choice of the (radial) cut-off functions u and a standard application of the co-area formula. This is part of the classical Federer-Fleming argument. Note that, using a rescaling procedure, the unit ball in (33) can be replaced by any ball of fixed radius $r > 0$. Obviously, in this case, the constant v will depend on r . We are now ready to prove the main result of the section.

Theorem 6.1. *Let $f : M \rightarrow \mathbb{M}^{m+1}(c)$ be a complete, non-compact, oriented, H -hypersurface with finite total curvature into the complete, simply connected space-form $\mathbb{M}^{m+1}(c)$ of constant sectional curvature $c \leq 0$. Then M satisfies the assumptions of Theorem 4.3, in particular, for every $1 < p < \infty$, the Calderón-Zygmund inequality $\text{CZ}(p)$ holds on M .*

Proof. Combining the Gauss equations with estimate (31) on the traceless second fundamental tensor, we deduce that M has bounded sectional curvature. On the other hand, M satisfies the non-collapsing condition (33). It follows from Theorem 4.7 in [11] that $r_{\text{inj}}(M) > 0$. Therefore, we can apply Theorem 4.3 above and conclude the validity of $\text{CZ}(p)$. \square

Remark 6.2. *The decay of the traceless second fundamental tensor of the H -hypersurface M of $\mathbb{M}^{m+1}(c)$ holds provided M has finite \mathbb{L}^p -total curvature $\|\Phi\|_p < \infty$ for some $m \leq p < \infty$, [29]. The conclusion of Theorem 6.1 can be extended accordingly.*

Remark 6.3. *As a matter of fact, inspection of the proof of Theorem 6.1 shows that it relies on two facts: (a) by Gauss equations, the sectional curvature of a manifold M is bounded if M is isometrically immersed, with bounded second fundamental form, into an ambient manifold of bounded curvature; (b) the injectivity radius of M is bounded from below by a positive constant provided M is isometrically immersed, with bounded mean curvature, into a Cartan-Hadamard manifold. Therefore, Theorem 6.1 can be extended in the following more abstract form:*

Theorem 6.4. *Let $f : M \rightarrow N$ be a complete Riemannian manifold isometrically immersed into a complete, simply connected manifold N with sectional curvatures satisfying $-A^2 \leq \text{Sec}_N \leq 0$. If the second fundamental tensor of the immersion satisfies $\|\mathbf{II}\|_\infty < \infty$, then M satisfies the assumptions of Theorem 4.3, in particular $\text{CZ}(p)$ holds on M , for every $1 < p < \infty$.*

APPENDIX A. HARMONIC COORDINATES

In this section, we collect some facts concerning harmonic coordinates. Let again $M \equiv (M, g)$ be an arbitrary smooth Riemannian m -manifold without boundary, let ∇ be the Levi-Civita connection and Δ the Laplace-Beltrami operator.

Definition A.1. *Let $x \in M$, $Q \in (1, \infty)$, $k \in \mathbb{N}_{\geq 0}$, $\alpha \in (0, 1)$. The $C^{k, \alpha}$ -harmonic radius of M with accuracy Q at x is defined to be the largest real number $r_{Q, k, \alpha}(x)$ with the following property: The ball $B_{r_{Q, k, \alpha}(x)}(x)$ admits a centered harmonic coordinate system*

$$\phi : B_{r_{Q, k, \alpha}(x)}(x) \longrightarrow \mathbb{R}^m,$$

(that is, $\phi(x) = 0$ and $\Delta\phi^j = 0$ on $B_{r_{Q,k,\alpha}(x)}(x)$ for each j), such that

(A.1)

$Q^{-1}(\delta_{ij}) \leq (g_{ij}) \leq Q(\delta_{ij})$ in $B_{r_{Q,k,\alpha}(x)}(x)$ as symmetric bilinear forms,

and for all $i, j \in \{1, \dots, m\}$,

(A.2)

$$\begin{aligned} & \sum_{\beta \in \mathbb{N}^m, 1 \leq |\beta| \leq k} r_{Q,k,\alpha}(x)^{|\beta|} \sup_{x' \in B_{r_{Q,k,\alpha}(x)}(x)} |\partial_\beta g_{ij}(x')| \\ & + \sum_{\beta \in \mathbb{N}^m, |\beta|=k} r_{Q,k,\alpha}(x)^{k+\alpha} \sup_{x', x'' \in B_{r_{Q,k,\alpha}(x)}(x), x' \neq x''} \frac{|\partial_\beta g_{ij}(x') - \partial_\beta g_{ij}(x'')|}{d(x', x'')^\alpha} \\ & \leq Q - 1. \end{aligned}$$

We shall refer to a coordinate system as above as a $\mathbb{C}^{k,\alpha}$ -harmonic coordinate system with accuracy Q on $B_{r_{Q,k,\alpha}(x)}(x)$.

Remark A.2. 1. Note that, when compared with the corresponding definition from [21], we additionally require $\phi(x) = 0$ here.

2. It is easily checked that the function $x \mapsto r_{Q,k,\alpha}(x)$ is globally Lipschitz.

3. By polarization, the inequality (A.1) implies that for some $C = C(m, Q) > 0$ it holds that

$$\max_{i,j \in \{1, \dots, m\}} \sup_{x' \in B_{r_{Q,k,\alpha}(x)}(x)} |g_{ij}(x')| \leq C,$$

in particular, putting everything together, there is a continuous decreasing function

$$F = F_{Q,k,\alpha,m} : (0, \infty) \longrightarrow (0, \infty),$$

such that the Euclidean $\mathbb{C}^{k,\alpha}$ -norm of the metric in this coordinates satisfies

$$(A.3) \quad \max_{i,j \in \{1, \dots, m\}} \|g_{ij}\|_{\mathbb{C}^{k,\alpha}} \leq F(r_{Q,k,\alpha}(x)).$$

This justifies the name $\mathbb{C}^{k,\alpha}$ -(harmonic) coordinate system.

Remark A.3. The natural differential operators of M (such as the gradient, the Laplacian and the Hessian of a given function) are defined in terms of the inverse metric coefficients g^{ij} . It is easy to see that, within the coordinate ball $B_{r_{Q,k,\alpha}(x)}(x)$, they inherit the $\mathbb{C}^{k,\alpha}$ -type

control, in terms of Q, m, α, k , of the metric coefficients g_{ij} . Indeed, the Cramer formula states that

$$(A.4) \quad g^{ij} = (-1)^{i+j} \frac{\det G_{ji}}{\det (g_{ij})},$$

where G_{ij} denotes the $(m-1) \times (m-1)$ matrix obtained from (g_{ij}) by deleting the i^{th} -row and the j^{th} -column. Both the numerator and the denominator of (A.4) are obtained as the sum of products of $C^{0,\alpha}$ -controlled functions and, by (A.1), $Q^{-m} \leq \det(g_{ij}) \leq Q^m$. Therefore, we can obtain a $C^{0,\alpha}$ control of the functions g^{ij} by using the $C^{0,\alpha}$ estimates of g_{ij} in combination with the following elementary fact:

Assume that $f, h : U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ satisfy $C^{-1} \leq h \leq C$ and $|f| \leq D$, for some constants $C, D > 0$. Then:

$$\begin{aligned} |\Delta_{x,y}(f/h)| &\leq C^3 |\Delta_{x,y}(f)| + C^2 D |\Delta_{x,y}(h)| \\ |\Delta_{x,y}(fh)| &\leq C |\Delta_{x,y}(f)| + D |\Delta_{x,y}(h)| \\ |\Delta_{x,y}(f+h)| &\leq |\Delta_{x,y}(f)| + |\Delta_{x,y}(h)|, \end{aligned}$$

where, to simplify the writings, we have set $\Delta_{x,y}(\bullet) = \bullet(y) - \bullet(x)$.

Now, differentiating the identity $g^{ik} \cdot g_{kj} = \delta_{ij}$ we get

$$(A.5) \quad \partial_t g^{ij} = -g^{ih} \cdot \partial_t g_{hj} \cdot g^{jk}.$$

It follows that a $C^{0,\alpha}$ control of ∂g^{ij} is obtained from those of g^{ij} and ∂g_{ij} . Proceeding inductively on the derivatives of (A.5) we finally deduce the claimed $C^{k,\alpha}$ estimate of g^{ij} .

The main result in this context states that control on the Ricci curvature up to order k together with control on the injectivity radius imply control on $r_{Q,k+1,\alpha}(x)$. To this end, for any $\Omega \subset M$ open and any $\varepsilon > 0$ let

$$\Omega_\varepsilon := \{x \mid x \in M, d(x, \Omega) < \varepsilon\} \subset M$$

be the ε -neighborhood of Ω .

Theorem A.4. *Let $Q \in (1, \infty)$, $\alpha \in (0, 1)$. Assume that there is an open subset $\Omega \subset M$, and numbers $k \in \mathbb{N}_{\geq 0}$, $\varepsilon > 0$, $r > 0$, $c_0, \dots, c_k > 0$ with*

$$|\nabla^j \text{Ric}(x)|_x \leq c_j, \quad r_{\text{inj}}(x) \geq r \quad \text{for all } x \in \Omega_\varepsilon, \quad j \in \{0, \dots, k\}.$$

Then there is a constant

$$C = C(m, Q, k, \alpha, \varepsilon, r, c_1, \dots, c_k) > 0,$$

such that for all $x \in \Omega$ one has $r_{Q,k+1,\alpha}(x) \geq C$.

Proof. Except the additional assumption $\phi(x) = 0$ that we have made for harmonic coordinates, this result can be found in [21] and the references therein (cf. Theorem 1.3 therein). However, since translations do not effect the required estimates, this is not a restriction. \square

In particular, the latter result implies $r_{Q,j,\alpha}(x) > 0$ for all $x \in M$, a fact which a priori is not clear at all. One calls the number

$$r_{Q,j,\alpha}(M) := \inf_{x \in M} r_{Q,j,\alpha}(x)$$

the $C^{k,\alpha}$ -harmonic radius for the accuracy Q .

APPENDIX B. GLUING RIEMANNIAN MANIFOLDS

Suppose we are given two Riemannian manifolds (M_1, g_1) and (M_2, g_2) with compact diffeomorphic boundaries and let $f : \partial M_1 \rightarrow \partial M_2$ be a fixed diffeomorphism⁶. The *Riemannian gluing* $\mathcal{M} = M_1 \cup_f M_2$ of M_1 and M_2 along f is the Riemannian manifold (\mathcal{M}, g) defined as follows.

As a topological manifold, \mathcal{M} is the quotient space obtained from the disjoint union $M_1 \sqcup M_2$ under the identification $x \sim f(x)$, for every $x \in \partial M_1$. It turns out that the natural inclusions $i_j : M_j \hookrightarrow \mathcal{M}$, $j = 1, 2$, are continuous embeddings.

Next, having fixed arbitrarily small collar neighborhoods $\alpha_j : \partial M_j \times [0, 2) \rightarrow M_j$ of ∂M_j , $j = 1, 2$, we consider the homeomorphism $\alpha : \partial M_1 \times (-2, 2) \rightarrow \mathcal{M}$ onto a neighborhood \mathcal{V} of $i_1(\partial M_1) = i_2(\partial M_2)$ defined as follows:

$$\alpha(x, t) = \begin{cases} i_1 \circ \alpha_1(x, -t) & t \leq 0 \\ i_2 \circ \alpha_2(f(x), t) & t \geq 0. \end{cases}$$

The original differentiable structures on M_1 and M_2 are then extended to a (unique up to diffeomorphisms) differentiable structure on \mathcal{M} by requiring that the natural inclusions i_j , $j = 1, 2$, are smooth embeddings and by pretending that α is a smooth diffeomorphism. See e.g. Chapter 8 of [22] and Chapter 5 of [26].

Finally, let $\mathcal{W} = \alpha(\partial M_1 \times (-1, 1))$, and fix any Riemannian metric g_3 on \mathcal{W} . For instance, we can pull-back on \mathcal{W} via α^{-1} a product metric $h + dt \otimes dt$ on $\partial M_1 \times (-1, 1)$. Let ξ_1, ξ_2 and ξ denote a partition of unity subordinated to the open covering $i_1(M_1 \setminus \partial M_1)$, $i_2(M_2 \setminus \partial M_2)$, and \mathcal{W} of \mathcal{M} . A Riemannian metric on \mathcal{M} is defined by setting

$$g = \xi_1 \cdot (i_1^{-1})^* g_1 + \xi_2 \cdot (i_2^{-1})^* g_2 + \xi \cdot g_3.$$

⁶different choices of f could produce non-diffeomorphic gluings as the exotic twisted spheres show.

Note that, outside the compact neighborhood $\overline{\mathcal{W}}$ of $i_1(\partial M_1) = i_2(\partial M_2)$, \mathcal{M} is isometric to the original open manifolds $M_j \setminus \alpha_j(\partial M_j \times [0, 1])$. In particular, different choices of g_3 will leave the corresponding Riemannian structure of \mathcal{M} in the same bilipschitz class. Moreover, if Ω is a domain compactly contained, e.g., in $M_1 \setminus \partial M_1$, then the collar neighborhood $\alpha_1(\partial M_1 \times [0, 2))$ of ∂M_1 can be chosen so to have empty intersection with Ω . Therefore, the neighborhood $\mathcal{V} \supset \overline{\mathcal{W}}$ of $i_1(M_1) = i_2(M_2)$ does not intersect $i_1(\Omega)$. Whence, it follows that Ω can be identified with its isometric copy $i_1(\Omega)$ into the glued space.

Now, if we specialize this construction to the case where $M_1 = M_2$ and $h = \text{id}$ we obtain “the” *Riemannian double* $\mathcal{M} = \mathcal{D}(M_1)$ of M_1 . On the other hand, if M_1 and M_2 are obtained by deleting a disk from the manifolds without boundaries \overline{M}_1 and \overline{M}_2 then we get the (rough and un-oriented) *Riemannian connected sum* $\overline{M}_1 \# \overline{M}_2$.

Acknowledgments. *The authors are indebted to Baptiste Devyver for pointing out the multiplicative inequality contained in Theorem 4.1 of [9]. Furthermore, the authors would like to thank Xiang-Dong Li and Feng-Yu Wang for a very helpful correspondence on the literature concerning Riesz-transforms.*

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